A LOGIC FOR THE ANALYSIS OF COLLATERAL ESTOPPEL

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The principle of collateral estoppel or issue preclusion[1] asserts that once an issue of ultimate fact[2] has been determined by a valid and final judgment in litigation, that issue cannot be relitigated between the same parties.[3] The principle's importance has increased greatly during the past fifteen years. An established rule of federal criminal procedure since 1916,[4] collateral estoppel became a constitutional requirement of due process enforceable against the states in 1970.[5] On the civil side, the determination that mutuality is not required for collateral estoppel to apply[6] and that the principle may be used offensively,[7] as well as the increase in mass tort cases,[8] have added to the principle's scope of

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1. The phrases "collateral estoppel" and "issue preclusion" are interchangeable. *See* F. JAMES & G. HAZARD, CIVIL PROCEDURE § 11.16, at 618 (3d ed. 1985) ("This . . . effect of a judgment was called 'collateral estoppel' and is now called 'issue preclusion'.").

2. Ultimate facts are facts necessary and essential for any determination or decision by the court. People *ex rel.* Hudson & M.R. Co., 44 N.Y.S.2d 884, 885 (1943).

3. Ashe v. Swenson, 397 U.S. 436, 443 (1970).

4. *Id.* (citing United States v. Oppenheimer, 242 U.S. 85 (1916)). For applications of collateral estoppel in federal criminal cases subsequent to 1916, see, e.g., Sealfon v. United States, 332 U.S. 575, 578 (1948); Yates v. United States, 354 U.S. 298, 335-38 (1957); Parklane Hosiery Co. v. Shore, 439 U.S. 322 *passim* (1979).

5. *See Ashe*, 397 U.S. at 436-37 (citing Benton v. Maryland, 395 U.S. 784 (1969), where the Court held that the fifth amendment guarantee against double jeopardy is enforceable against the states through the fourteenth amendment).

6. *See* Blonder-Tongue Labs., Inc. v. University of Ill. Found., 402 U.S. 313 (1971). Mutuality required that, unless both parties in a second action were bound by a judgment in a previous case, neither could be; that is, collateral estoppel applied only when both parties to the second suit were parties to, or in privity with parties to, the earlier suit in which the issue was litigated and determined. *Id.* at 320-21. *See also* Bernhard v. Bank of Am., 19 Cal.2d 807, 122 P.2d 892 (1942) (earlier state rejection of the mutuality requirement by the California Supreme Court).

7. *See* Parklane Hosiery Co. v. Shore, 439 U.S. 322 (1979). *Parklane* also discusses the appropriate circumstances for the use of offensive collateral estoppel, i.e., the use by a second plaintiff of a fact found in an earlier suit against the same defendant. *See also* United States v. United Air Lines, Inc., 216 F. Supp. 709 (E.D. Wash. & D. Nev. 1962) (earlier case allowing offensive use under some circumstances), *aff'd sub nom.* United Air Lines, Inc. v. Wiener, 335 F.2d 379 (9th Cir. 1964).

8. In 1982 there were

> pending in federal and state courts over 16,000 asbestos-related lawsuits, over 1,500 "Dalkon Shield" cases, at least 1,000 DES cases, nearly 1,000 cases alleging separate design defects against two car manufacturers, over 500 cases stemming from exposure to formaldehyde foam insulation, several hundred Agent

application.

If collateral estoppel applied only to judgments, its application would be straightforward; however, it applies not only to the judgment, but also to all issues essential to that judgment,[9] which raises the problem of determining just what facts were found in the earlier proceeding. It is a determination that is often difficult.

> [I]n criminal cases tried before a jury . . . [t]he precise determination of what a jury decided and why can be particularly difficult. In non-jury cases an appellate court is presented with the decision of a trial judge and an explanation of his reasoning. But non-jury verdicts are the exception rather than the rule.[10]

Special findings of fact, which are common in a civil trial, facilitate the identification of the issues that were litigated. Furthermore, if the matter was tried before a jury, jury interrogatories or a special verdict may prove helpful. But if the jury determined a general verdict, ascertaining the precise issues determined may be difficult.[11]

The Supreme Court has stated that "the rule of collateral estoppel . . . is . . . to be applied . . . with realism and *rationality*,"[12] but the informal analysis used by appellate courts leads to little certainty in the establishment of precisely what was determined. Often the conclusion is simply

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> Orange lawsuits, and hundreds of other cases involving exposure to various chemical products. In addition, scores of cases have been filed by plaintiffs who were injured or had relatives that were killed in plane, train or bus accidents.

Gunn, *The Offensive Use of Collateral Estoppel in Mass Tort Cases*, 52 MISS. L.J. 765, 765-66 (1982) (footnotes omitted).

9. *See* RESTATEMENT (SECOND) OF JUDGMENTS § 68(j) (tent. draft no. 4 1977).

10. United States v. Standefer, 610 F.2d 1076, 1095 (3d Cir. 1979), *aff'd*, 447 U.S. 10 (1980). *See also* Vestal, *Issue Preclusion and Criminal Prosecutions*, 65 IOWA L. REV. 281, 291 (1980) (recognizing the general difficulty of ascertaining what issues have been decided in making the general finding where the first case was a criminal prosecution, but also noting that in such a situation a determination may be possible either from an examination of the evidence presented or where the court has found facts specially, as under FED. R. CRIM. P. 23(c)).

11. Vestal, *supra* note 10, at 291. *See also* Note, *Invoking Collateral Estoppel Offensively: The Ends of Justice . . . or the End of Justice?*, 4 AM. J. TRIAL ADVOC. 75, 87 (1980) ("There also exists a real possibility that the verdict in the former adjudication was the result of a jury compromise. . . . When collateral estoppel is then asserted the effect is to require subsequent courts to accept as decided, issues which may not have, in fact, been decided.") (footnotes omitted).

Seemingly contrary on the issue of prior civil determinations' being somewhat less difficult than criminal ones is Thau, *Collateral Estoppel and the Reliability of Criminal Determinations: Theoretical, Practical, and Strategic Implications for Criminal and Civil Litigation*, 70 GEO. L.J. 1079, 1114 (1982) ("Determining the findings essential to criminal cases may be easier than for civil cases because criminal offenses tend to be exclusively statutory.") (footnote omitted). But, Thau was considering the use of criminal *convictions* in later civil cases. Thau also admits, however, that "such determinations can be difficult, if not impossible, to prove in some cases." *Id.*

12. *Ashe*, 397 U.S. at 444 (emphasis added).

a bald assertion, perhaps reflecting more the appellate judge's view of the facts (or the view of the second trial judge) than a reasoned determination of the jury's view and leaving a great deal of room for disagreement.[13]

At least part of the problem with the absence of much formal analysis may be due to the lack of an applicable system of logic with which to structure the reasoning. Courts have been instructed that to determine whether collateral estoppel applies they are to "examine the record of the prior proceeding, taking into account the pleadings, evidence, charge, and other relevant matter, and conclude whether a rational jury could have grounded its verdict upon an issue other than that which the [party] seeks to foreclose from consideration."[14] However, currently available logic provides no guidance in that determination.

Treating findings of fact as true propositions in standard propositional logic is inadequate. In addition to epistemological objections over whether the findings are true, there are more serious logical problems. Findings of fact are probability based,[15] while the propositional calculus is two-valued. When findings are combined, the inappropriateness of propositional logic becomes apparent.

As an example, consider a case in which there have been findings of *p*, *p implies q*, *q implies r*, *r implies s*, . . . , and *y implies z*. If each finding is taken as a true logical proposition, the truth of *z* follows logically from the findings; however, *p* and each of the implications has been established probabilistically. While there may be sufficient probability for *p* and *p implies q* each to be taken as a finding, *q* may not be sufficiently probable. Since *q* only probably follows from *p*, its probability may be less than that of *p*. As the chain of implications is followed, the probability of the conclusions may eventually drop below the level required for a finding of fact. The propositional calculus derives conclusions equal in strength to the propositions on which they are based, but where probabilistically based claims are combined, the

13. *Compare, e.g.*, *Ashe*, 397 U.S. at 445 ("The single rationally conceivable issue in dispute before the jury was whether the petitioner had been one of the robbers. And the jury by its verdict found that he had not.") *with* 397 U.S. at 467 (Burger, C.J., dissenting) ("The majority's analysis of the facts disregards the confusion injected into the case by the robbery of Mrs. Gladson. . . . [T]he jury could well have acquitted Ashe yet believed that he was present in the Gladson home.").

14. Sealfon v. United States, 332 U.S. 575, 579 (1948).

15. *See* MCCORMICK ON EVIDENCE 957 (3d ed. 1984) ("The most acceptable meaning to be given to the expression, proof by a preponderance, seems to be proof which leads the jury to find that the existence of the contested fact is more probable than its nonexistence.").

strengths of the conclusions drop.[16] An analysis that adequately handles findings of fact must account for the probabilistic strength of the propositions involved.

A situation in which there have been findings of *p* and *q* presents a second example. If *p* and *q* are taken as true logical propositions, the truth of the conjunction, *p and q*, would follow. Such should not be the case with findings. Findings are probabilistically based, and the probability of a conjunction is less than or equal to the probability of either conjunct.[17] So while *p* and *q* may each be sufficiently probable to be taken as found, the probability of the conjunction may fall below the threshold level for a finding of fact.[18]

The problems with propositional logic seem to indicate the need for a probability-valued logic or for the assignment of probabilities to findings[19] and the mathematical calculation of the probabilities of related propositions. Both of these alternatives, however, have their costs. First, each would require the abandonment of two-valued logic—a logic that has proven to be a powerful tool in the analysis of arguments in many areas; second, neither alternative can comprehend the possibility of a proposition being both probable and false.[20] While there has been debate over whether a proposition can be probable but false (or improb-

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16. The problem is similar to one noted in L. COHEN, THE PROBABLE AND THE PROVABLE 68-73 (1977).

17. For example, the probability of an integer between one and ten, inclusive, selected at random being even is one-half. The probability of that integer being greater than five is also one-half. The probability of its being even and greater than five is three-tenths. Generally, requiring the selected element to have two characteristics cannot result in the selection of more elements than requiring the selected element to have one, but not necessarily the other, characteristic.

18. Similar problems would result with findings of *p or q* and *not-p*. Treating the findings propositionally would allow the inference of *q*. Such an inference should not hold for probabilistically based propositions. If an integer is selected at random from among 1, 2, 3, it is probable that 2 or 3 was selected. It is also probable that 2 was not selected. But, it is not probable that 3 was selected.

19. There are difficulties in the assignment of probabilities in legal analysis. *See generally* L. COHEN, *supra* note 16. The analysis envisioned here, however, would be employed only after there have been some findings. Those findings might then be assumed to have some probability value greater than *k*, where *k* is some selected level greater than or equal to one-half but less than one (greater than one-half in cases involving a higher standard of proof). An algebraic calculation, using the laws of probability, would then identify other logically related propositions whose probabilities are also greater than *k*.

20. The probability assigned to a false proposition would have to be close to (or equal to) zero, but a proposition that is probable would have a probability greater than one-half. Since assigning a probability value is the only way to express "true," "false," "probable," or "improbable" in a probability-valued logic or probability calculus, expressing the fact that a proposition is probable but false requires the assignment of two inconsistent probability values. Improbable but true propositions present a similar problem.

able but true),[21] there are easily constructed examples showing that such propositions exist. Such an example results from the roll of a die, where the outcome is not seen. It is probable that there are more than two spots on the upper face, but in roughly one-third of the repetitions it will also be false.

Similarly, it may be desirable for a logical analysis to be capable of interpreting a situation in which a proposition has been found as fact but is actually false. If a fact-finder is anything less than perfect, such situations will occur. Indeed, the imposition of a burden of proof creates the possibility of a proposition not found as fact being true.

Admittedly, it is very unlikely that a collateral estoppel argument will require the consideration of a situation in which a proposition found as fact is false, since once found as fact, it cannot be argued to be false in a collateral proceeding. Nevertheless, if an analysis that can comprehend such a situation may be had at little or no cost, it would seem preferable to an analysis that logically requires the truth of a proposition that has been found as fact. The cost of the system to be developed here is the inability to express precise probability values. The system can only express the fact that the probability either exceeds or fails to exceed some value greater than or equal to one-half. But, this is not a major loss, since that is all that is indicated by a finding of fact at trial. Furthermore, the logic to be developed here will have application in the analysis of other areas involving findings of fact, and in those areas the combined analysis of findings, true or false propositions and conceptually necessary or possible propositions may be useful.

Part I of this article examines how a logic may be developed that can comprehend probabilistically based propositions within a two-valued framework, thus maintaining the possibility of a proposition being found as fact and also being false. In Parts II and III such a logic will be developed. Lastly, Part IV presents examples of the application of this logic.

I. A MODAL TREATMENT OF JURY DELIBERATIONS

An intuitive explanation of the logic to be developed here may be found in the possible worlds description of modal logic. Modal logic[22] is the logic of necessity and possibility. It recognizes that some proposi-

21. *Compare* Toulmin, *Probability, reprinted in* ESSAYS IN CONCEPTUAL ANALYSIS 157 (A. Flew ed. 1956) *with* King-Farlow, *Toulmin's Analysis of Probability*, 29 THEORIA 12 (1963).

22. *See generally* C. LEWIS & C. LANGFORD, SYMBOLIC LOGIC (1932); G. VON WRIGHT, AN ESSAY IN MODAL LOGIC (1951).

tions *must* be true, while others *may* be true or false and still others *cannot* be true.[23] The possible worlds analysis considers all the possible worlds or states of affairs that are alternatives to the actual.[24] The necessary propositions are those that must be true under every alternative. Propositions are possible, if they are true under some alternative, and impossible, if they are true in no alternative world.

The treatment of probability in the possible worlds explanation rests on interpreting it as the ratio of the number of possible worlds in which the proposition is true to the totality of possible worlds.[25] In each

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23. There are several bases for a claim that a proposition *must* be true. The proposition may be tautological (logically necessary), necessary within the context of a theory (theoretically necessary) or conceptually necessary. *See* D. SNYDER, MODAL LOGIC AND ITS APPLICATIONS 167-78 (1971). Similarly, propositions may be self-contradictory (logically impossible), impossible within the context of a theory (theoretically impossible) or inconceivable (conceptually impossible).

24. For a development of the possible worlds semantics, see Kripke, *Semantic Analysis of Modal Logic I: Normal Modal Propositional Calculus*, 9 ZEITSCHRIFTE FÜR MATHEMATISCHE LOGIK UND GRUNDLAGEN DER MATHEMATIK 67 (1963).

Kripke's development starts with model structures, which are ordered triples (G, W, R), where W is a set, G is an element of W, and R is a reflexive relation (a relation in which everything relates to itself and possibly to others as well; that is, for every H, H relates to H (written *HRH*)) on W. W may be interpreted as the set of all possible worlds or equiprobable states of affairs, G as the actual world or actually existing state of affairs and R as the relation "has access to" or "is able to examine."

Models on these structures are binary functions, f(p,H), which assigns a truth value 0 or 1 for each proposition, p, and each world, H, belonging to W. Truth values for compound formulae are assigned as follows:

If f(p,H) = f(q,H) = 1, then f(p&q,H) = 1, where *p&q* represents the conjunction of p and q; otherwise, f(p&q,H) = 0.

If f(p,H) = 1, then f(~p,H) = 0, where *~p* represents the negation of p; otherwise, f(~p,H) = 1.

f(pvq,H) = 1, if f(p,H) = 1 or f(q,H) = 1, where *pvq* represents the disjunction of p and q; otherwise, f(pvq,H) = 0.

f(p→q,H) = 0, if f(p,H) = 1 and f(q,H) = 0, where *p→q* represents the material implication of q from p; otherwise, f(p→q,H) = 1.

f(p↔q,H) = 1, if f(p,H) = f(q,H), where *p↔q* represents material equivalence; otherwise, f(p↔q,H) = 0.

f(Lp,H) = 1, if f(p,H′) = 1 for each H′ such that H has access to H′ (i.e., HRH′), where *Lp* represents "*p* is necessary"; otherwise, f(Lp,H) = 0.

f(Mp,H) = 1, if f(p,H′) = 1 for some H′ such that HRH′, where *Mp* represents "*p* is possible"; otherwise, f(Mp,H) = 0.

Kripke took R to be a reflexive relation, and (G, W, R) and f then provided a model for Feys' modal system T, *see* Feys, *Les Logiques Nouvelles des Modalites*, 40 REVUE NEOSCHOLASTIQUE DE PHILOSOPHIE 517 (1937), or von Wright's equivalent system M. *See* G. VON WRIGHT, *supra* note 22. If *R* is also transitive (if ARB and BRC, then ARC), then a model for Lewis' S4 results. *See* C. LEWIS & C. LANGFORD, *supra* note 22, at 492-502. Adding symmetry (if ARB, then BRA) to reflexivity and transitivity yields a model for Lewis' S5. *See id.* On the other hand, dropping the reflexive property in favor of an existence requirement (there is some H′ such that HRH′) yields a model for the logic Mn. *See* G. VON WRIGHT, *supra* note 22.

25. To interpret probability values within the formal model we will restrict the order

world the proposition is either true or false and the system as a whole is two-valued, yet probability values may be interpreted. If the probability exceeds some set value greater than or equal to one half, the proposition is probable (or may be taken as a finding of fact).[26] It may be so designated in the logical system and the relation of the probable statements (or findings of fact) to other propositions may be examined within a two-valued context. Such a treatment is consistent with the position developed earlier with regard to the probable proposition (or the proposition found as fact) and truth.[27] While there is an implication of the truth of a proposition from its necessity and of possibility from truth, there is no such logical connection between the probable (or findings) and truth. The fact that a proposition is true in more than some fixed proportion of worlds does not guarantee that the proposition is true in any particular world, including the actual. Nor does its truth in the actual world guarantee its truth in the required proportion of worlds.

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of W in the model structure so that W is finite. *See supra* note 24. For each H in W we will let J(H) be the set of all H′ in W to which H has access; i.e.,

$J(H) = \{H' \mid H' \in W \text{ and } HRH'\}$.

Given a proposition, p, with a probability value, v, in world H, assign truth values 0 or 1 to p in each world H′ in J(H) so that the sum of the f(p,H′)s over J(H) equals v; that is, so that

$$\frac{\sum_{H' \in J(H)} f(p,H')}{n} = v.$$

Note that $n \neq 0$, since under either the reflexive or the existence property there is at least one world in J(H).

There are shortcomings to this approach. First, given the limitation on the number of worlds in which truth values may be assigned, probability values may need to be approximated. Secondly, a finite number of worlds can model only a finite number of propositions. While more mathematically sophisticated methods might be employed to remedy these problems, the problems are actually of no concern here. Given the inexact probabilities involved in collateral estoppel analysis and the limited number of propositions of interest in any given analysis, the model presented is adequate.

26. The Kripke model, *see supra* note 24, is extended by introducing *Fp* which is interpreted as *p is probable* or *p has been found in fact.* Hereinafter writing

$$\frac{\Sigma}{H' \in J(H)} \text{ as } \Sigma, \text{ we let}$$

$$f(Fp,H) = 1, \text{ if } \frac{\Sigma f(p,H')}{n} > k; \text{ otherwise, } f(Fp,H) = 0.$$

K must be greater than or equal to one-half and in legal applications is determined by the evidentiary level required (although the probability levels represented by "clear and convincing" and "beyond a reasonable doubt" are unclear). K will be taken as one-half in all further discussion of the model, but this does not result in any loss of generality. The arguments to be presented could be reconstructed using any value for k greater than one-half but less than one.

27. *See supra* notes 19-21 and accompanying text.

The interpretation also preserves the fact that a proposition and its negation cannot both be probable. The proportion of worlds in which p is true and the proportion in which its negation $\sim p$ is true cannot both exceed one half, since a proposition and its negation cannot both be true in any particular world. Carried over to the legal interpretation, a jury may not find as fact both a proposition and its negation.

Another property preserved by the model is that it may occur that neither p nor $\sim p$ is probable, as where each is true in exactly half the worlds that may be examined. Therefore, from the fact that p is not probable one cannot infer that $\sim p$ is probable. In the trial process the burden of proof will determine the outcome in such a case; however, it would not seem correct to claim that the burden has determined any factual findings. It simply determines the conclusion, when the jury has failed to find either p or $\sim p$.

Despite the possibility that in the model neither p nor $\sim p$ is found, there are situations (where a presumption exists) in which one alternative must be found. Since it is not true in the model that either Fp (p is found) or $F\sim p$ ($\sim p$ is found) must be true, presumptions will not arise as a result of the logic. Rather, if a presumption is involved in a case, it must be taken as a supposition in the particular analysis. If there is a rebuttable presumption p, it will be expressed as the implication: "if $\sim p$ is not found, then p is found."[28] An irrebuttable presumption p will be taken as the supposition that p is necessary (Lp); that is, p must be taken as true under any alternative state of affairs. Since jury instructions must also be applied to any alternative, they too will be taken as expressing necessary propositions.

Before turning to the actual development of the logical systems, a word should be said about differences in evidentiary strength. So long as the level above which a proposition is taken as found as fact (i.e., Fp is taken as true) is greater than or equal to one-half (but less than one), the logic will not change with changes in the level. An analysis involving findings by a preponderance will be the same as it would be if the findings were all by clear and convincing evidence or were found beyond a reasonable doubt. Difficulties may arise if findings of differing strengths are involved in one analysis;[29] however, if all the findings are

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28. Other presumptions may take the form "if p is found, then q is found." While presumptions of this form may well be more common and as a result more interesting in application, the form discussed in the text is more interesting in the development of the logic, since it provides the only instance in which a finding of p can be inferred from a failure to find *not-p*, and *not-p* from a failure to find p.

29. Note that this would also be a problem in the algebraic calculation of probability

reduced to the lowest evidentiary level involved, conclusions at that lowest level may be derived.

We may now turn to the systems themselves. Since possible worlds analysis provides models for several modal logics, the extensions of those models will aid in the development of modal systems, incorporating those propositions found as fact that are extensions of those logics. The reasonableness of axioms may be checked against the models and the models will provide a source of counter-examples to show that certain formulae should not be theses.[30] The development will proceed on two levels. In the text the systems will be presented in a rigorous but non-formal manner. The formality that is absent in the text may be found in the accompanying notes and in the appendices.

II. THE SYSTEM FT

The logic FT is developed as an extension of the modal logic T.[31] It is obtained by appending additional primitive symbols, formation rules and axioms to those of T.

The primitive symbols include variables representing propositions:

$p, q, r, \ldots;$

unary operators representing necessity, negation and finding as fact:

$L, \sim, F;$

and a binary connective representing disjunction (the non-exclusive *or*):

v.

From these primitive symbols certain formulae, known as well-formed formulae, may be constructed using combinations of the following formation rules:

1. A variable standing alone is a well-formed formula.
2. If α is a well-formed formula, then so are $L\alpha$, $\sim\alpha$ and $F\alpha$.
3. If α and β are well-formed formulae, then so is $(\alpha v \beta)$. Where no ambiguity results, parentheses may be omitted.

There are also defined symbols representing conjunction, material implication, logical equivalence and possibility:

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values. Since those values too are inexact, but merely exceed some value, calculations involving lesser evidentiary strength will not lead to conclusions at higher levels.

30. Checking the axioms against the model also provides a relative consistency proof for the logic. The model is based in set theory, so interpreting the axioms in the model shows that they will not lead to any contradictions not already present in set theory. The use of the model to show that certain formulae should not be theses falls far short of a completeness proof and is not intended as such.

31. *See* Feys, *supra* note 24. The system T is equivalent to the modal system M, developed by von Wright. *See* G. VON WRIGHT, *supra* note 22.

&, →, ↔, and M.[32]

&, → and ↔ follow rule three; that is, if α and β are well-formed formulae, then so are (α & β), (α→β) and (α↔β). *M* follows rule two; that is, if α is a well-formed formula, then Mα is as well.

It should be noted that, under the symbols and rules adopted, any well-formed formula of the standard propositional logic or of T will be a well-formed formula of FT. In addition, an axiomatic basis for propositional logic and for T, and transformation rules adequate for each, will be included in FT. Therefore, any thesis of the propositional calculus or of T will be a thesis of FT.

The axioms of FT consist of the following:

A 1 (pvp)→p — If *p or p* is true, then *p* is true.

A 2 p→(pvq) — If *p* is true, then *p or q* is true.

A 3 (pvq)→(qvp) — If *p or q* is true, then *q or p* is true.

A 4 (q→r)→((pvq)→(pvr)) — If *q* implies *r*, then (*p or q*) implies (*p or r*).

A 5 Lp→p — If *p* is necessary, *p* is true, or alternatively, if *p* is true in every world, *p* is true in the actual world.

A 6 L(p→q)→(Lp→Lq) — If it is necessary that *p* implies *q*, then if *p* is necessary so is *q*, or alternatively, if *p* implies *q* in every world, then if *p* is true in every world *q* is true in every world.

A 7 Lp→Fp — If *p* is necessary, it is probable or must be found as fact. Alternatively, if *p* is true in every world, it is true in more than the needed proportion of worlds.

A 8 ~(Fp & F~p) — *p* and ~*p* cannot both be found as fact (within the same analysis).

A 9 F(p & q)→(Fp & Fq) — If (*p and q*) is found as fact, *p* can be taken as found, as can *q*.

A 10 (Lp & F (p→q))→Fq — If *p* is necessary and (*p implies q*) is found as fact, *q* may be taken as found.

A 11 (Fp & L(p→q))→Fq — If *p* is found as fact and *p* necessarily implies *q*, *q* may be taken as found.

Axioms A 1 through A 4 provide an axiomatic basis for propositional logic, and the addition of A 5 and A 6 completes a basis for T.

Axioms A 7 through A 11 are additional axioms for the operator *F*; that is, they incorporate findings of fact into the modal system. An examination of the possible worlds model would show that the axioms are reasonable choices to govern the use of *F*, if *F* is to be interpreted as indicating a probabilistically based legal finding.[33]

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32. $\alpha \& \beta =_{def} \sim(\sim\alpha v \sim\beta)$
$\alpha \rightarrow \beta =_{def} \sim\alpha v \beta$
$\alpha \leftrightarrow \beta =_{def} (\sim\alpha v \beta) \& (\sim\beta v \alpha)$
$M\alpha =_{def} \sim L \sim\alpha$

33. To see that A 7 holds in the model assume f(Lp,H) = 1. Therefore,

All that remains to complete the basis for the logic is to add transfor-

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$$\frac{\Sigma f(p,H')}{n} = \frac{n}{n} > \frac{1}{2}.$$

So f(Fp,H) = 1 and *Lp→Fp* holds in the model. Put more simply, if *p* is true in every world, it must be true in at least half the worlds.

Next consider Axiom A 8. For any p and any H either f(p,H) = 1 or f(p,H) = 0, in which case f(∼p,H) = 1. If it is assumed that f(Fp,H) = 1 = f(F∼p,H), that is in some world, H, both *p* and *∼p* are found, then:

$$\frac{\Sigma f(p,H')}{n} > \frac{1}{2} \text{ and } \frac{\Sigma f(\sim p,H')}{n} > \frac{1}{2}.$$

Therefore,

$$\frac{\Sigma f(p,H')}{n} + \frac{\Sigma f(\sim p,H')}{n} > 1,$$

and since each summation is over J(H) for the same H, then:

$$\frac{\Sigma f(p,H') + \Sigma f(\sim p,H')}{n} > 1.$$

However, since for each H′, one of f(p,H′) and f(∼p,H′) equals one and the other zero, then:

$$\frac{\Sigma f(p,H') + \Sigma f(\sim p,H')}{n} = \frac{n}{n} = 1.$$

This leads to 1 > 1, a contradiction with the rules of arithmetic, so *∼(Fp&F∼p)* holds in the model.

To show that Axiom A 9 holds in the model assume f(F(p&q),H) = 1. Then:

$$\frac{\Sigma f(p\&q,H')}{n} > \frac{1}{2}.$$

However, whenever f(p&q,H′) = 1, f(p,H′) = 1, that is in any wold in which *p&q* is true, *p* is true, so f(p,H′) ≥ f(p&q,H′) for each H′. Therefore

$$\frac{\Sigma f(p,H')}{n} \geq \frac{\Sigma f(p\&q,H')}{n} > \frac{1}{2},$$

so f(p,H) = 1. The argument for Fq is the same, so *F(p&q) (Fp&Fq)* holds in the model.

To show that A 10 holds assume f(Lp&F(p →q),H) = 1, so f(Lp,H) = 1 = f(F(p→q),H). Therefore, f(p,H′) = 1 for each H′ in J(H) and:

$$\frac{\Sigma f(p\rightarrow q,H')}{n} > \frac{1}{2}.$$

For each H′ in which *p→q* has truth value 1, so does *p*. Since the propositional logic is assumed in each element of W, *q* will have truth value 1 in each H′ in which *p→q* has truth value 1. Also, if *q* has truth value 1 in an H′, so does *p→q*. Hence:

$$\frac{\Sigma f(q,H')}{n} = \frac{\Sigma f(p\rightarrow q,H')}{n} > \frac{1}{2}.$$

mation rules—rules that provide for the inference of one proposition from another (or others). The following rules are adequate for the systems developed here:

1. Uniform Substitution — If any variable in a thesis is uniformly replaced by any well-formed formula, the result will be a thesis.
2. Modus Ponens — If α and (α→β) are theses, so also is β. (If α can be proved and so can (α→β), then β may be taken as proved.)
3. Necessitation — If α is a thesis, so is Lα. (If α can be proved, it is necessary.)

There are also rules which may be derived from the basis thus far presented. They are unnecessary in that anything that may be proved using the derived rules may be proved without them, but they are useful enough to justify their inclusion as rules.

4. Conjunctive Simplification (CS) — If (α&β) is a thesis, then so are α and β. (The proof of (α and β) is sufficient to prove α and to prove β).
5. Conjunctive Inference (CI) — If α and β are theses, so is (α&β). (The proof of α and of β is sufficient to prove (α&β).)
6. Inference by Cases (IC) — If (αβ), (α→γ), and (β→γ) are theses, so is γ. (If (α or β) has been proved and each implies γ, γ is proved.).

The deduction theorem also holds in systems using the transformation rules given here.[34] Under that theorem, if one is attempting to prove the conditional (α→β), it may be done by taking α as a presupposition and proving β. Since indirect inference may be derived from the deduction theorem in a propositional calculus basis, it too will be allowed. Indirect inference allows the proof of α (or ∼α) by showing that the assumption of ∼α (or α) leads to a logical contradiction.

One last rule is to accept the substitution of equivalent well-formed

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Therefore, f(Fq,H) = 1 and *(Lp&F(p→q))→Fq* holds in the model.

Lastly, consider Axiom A 11 and assume f(Fq&L(p→q),H) = 1. So, f(Fp,H) = 1 = f(L(p→q),H). Therefore, f(p→q,H′) = 1 for each H′ in J(H) and:

$$\frac{\Sigma f(p,H')}{n} > \frac{1}{2}.$$

For each H′ in which *p* has the truth value 1, *p→q* also has truth value 1, so *q* will also have truth value 1. Therefore:

$$\frac{\Sigma f(q,H')}{n} \geq \frac{\Sigma f(p,H')}{n} > \frac{1}{2},$$

so f(Fq,H) = 1 and *Fp&L(p→q))→Fq* holds in the model.

34. *See* Zeman, *The Deduction Theorem in S4, S4.2 and S5*, 8 NOTRE DAME J. OF FORMAL LOGIC 56 (1967).

formulae into any formula that is not a finding.[35] Where the formula within which the substitution is to take place is a finding, the substituted formula must be necessarily equivalent to the formula it replaces.[36]

We are now ready to examine several theorems of FT.[37] The first seven are theorems of T that will tell us nothing of findings directly but will be useful in the examination of findings.

Theorem 1: $L(p \& q) \leftrightarrow (Lp \& Lq)$: If (*p and q*) is necessary, each of *p* and *q* is individually necessary, and vice versa.

Theorem 2: $Lp \leftrightarrow \sim M \sim p$: The necessity of *p* is equivalent to the impossibility of *not-p*.

Theorem 3: $M(pvq) \leftrightarrow (MpvMq)$: If (*p or q*) is possible, either *p* is possible or *q* is possible, and vice versa.

Theorem 4: $M(p \& q) \rightarrow (Mp \& Mq)$: If (*p and q*) is possible, then each singly is possible.

Theorem 5: $Lp \rightarrow L(q \rightarrow p)$: If *p* is necessary, then *q* will necessarily imply *p*. Alternatively, if *p* is true in every alternative, then in every alternative *q* implies *p*.

Theorem 6: $(LpvLq) \rightarrow L(pvq)$: If either *p* is necessary or *q* is necessary, (*p or q*) is necessary. Alternatively, if *p* is true in every world or *q* is true in every world, *p or q* is true in every world.

Theorem 7: $L(p \rightarrow q) \rightarrow (Mp \rightarrow Mq)$: If (*p implies q*) is necessary, then if *p* is possible so is *q*. Alternatively, if in every world *p* implies *q*, then if *p* is true in some world *q* is true in some world (in fact the same world in which *p* was true).

The remaining theorems do involve findings of fact.

Theorem 8: $Fp \rightarrow F(pvq)$: If there is a finding of *p*, then (*p or q*) may be taken as having been found.

Theorem 9: $F \sim p \rightarrow \sim Fp$: If *not-p* has been found, *p* cannot also be found as fact.

Theorem 10: $Fq \rightarrow F(p \rightarrow q)$: If *q* is found, (*p implies q*) may also be taken as found.

Theorem 11: $F(p \& q) \rightarrow F(pvq)$: If (*p and q*) is found, (*p or q*) may be taken as found.

Theorem 12: $(F(pvq) \& L \sim p) \rightarrow Fq$: If (*p or q*) has been found but

35. For a proof that the substitution of equivalents holds in formulae built using ~, v and *L* as connectives, see G. HUGHES & M. CRESSWELL, AN INTRODUCTION TO MODAL LOGIC 33-37 (1972).

36. This need not be assumed as a rule but can be proved from Axioms A 10 and A 11 and from Theorem 1, which requires the substitution of equivalents in its own proof but not within the scope of an "F".

37. Proofs of the theorems stated in the text may be found in Appendix 1, *infra*. Since the first six theorems are theorems of T, their proofs are not presented. They may be found in G. HUGHES & M. CRESSWELL, *supra* note 35, at 34-40.

	the negation of *p* is conceptually necessary; i.e., *p* is inconceivable, then *q* may be taken as found.
Theorem 13:	$\sim$F(pvq)$\rightarrow$($\sim$Fp & $\sim$Fq): If (*p or q*) has not been found, then neither *p* nor *q* has been found.
Theorem 14:	L(p$\rightarrow$q)$\rightarrow$(Fp$\rightarrow$Fq): If (*p implies q*) is conceptually necessary, then if *p* is found *q* is found.
Theorem 15:	(Lp & Fq)$\rightarrow$F(p & q): If *p* is conceptually necessary and there is a finding that *q*, then (*p and q*) may be taken as found.
Theorem 16:	Fp$\rightarrow$Mp: If *p* is found, *p* must be conceivable.
Theorem 17:	p$\rightarrow$Mp: If *p* is true, *p* is possible (or conceivable).

III. THE SYSTEM FS4

One difficulty with the logic presented is that iterations of modalities may result through the operation of the axioms and theorems of FT. For example, it may be possible for both *p* and *q* to be found; that is, *M(Fp & Fq)* may be true. The operation of Theorem 4 on this formula yields *M F p & M F q*, which by conjunctive simplification will result in either *M F p* or *M F q*. *M F p* and *M F q* are examples of iterated modalities—a string of modalities attached to a well-formed formula. The interpretations of these particular iterations make sense within the system. The first would be interpreted as "It is possible that *p* be found." Indeed, this is the claim made by one opposing a motion for summary judgment.[38] Other iterations, for example *M L F Mp* or even *F F p* make less or no sense under our interpretation of the symbols involved. However, such iterations may arise in proofs. To handle this situation a logic must be developed to allow the implication of a less complex iteration from a string of modalities. FS4 is such a logic.

FS4 is an extension of FT that parallels the extension in modal logic of T to S4. The T to S4 extension is accomplished by adding the axiom *Lp* $\rightarrow$ *LLp*—if *p* is necessary, it is necessary that *p* is necessary. The additional axiom may seem uninteresting, but it does provide for the reduction of strings of iterated modalities[39] that do not involve *F*. An FS4 axiom must also be added if iterations involving *F* are also to be reduced.[40]

38. *See* FED. R. CIV. P. 56; F. JAMES & G. HAZARD, *supra* note 1, § 4.10, at 206, and § 5.19, at 272-73.

39. There are no infinite strings of modalities in S4. Any string of modalities reduces to one of the following: p, Lp, Mp, LMp, MLp, LMLp, MLMp. *See* G. HUGHES & M. CRESSWELL, *supra* note 35, at 45.

40. The FS4 axiom will allow the implication of a shorter string of modalities from a string containing an F. Given the collapse of S4 iterations, *see* note 39 *supra*, what must be considered are: MF, FM, FL, LF, FF, LMF, MLF, LMLF, MLMF. They are considered *infra* notes 43-45 and accompanying text. The other iterations—FLM, FML, FLML

In terms of the possible worlds model, the change to S4 (or FS4) is matched by the addition of the transitive property to the accessibility relation.[41] If the actual world has access to examine an alternative and the alternative has access to examine a second alternative, the actual also has direct access to the second alternative.[42] The reduction in the number of steps required in the examination of alternatives leads to the reduction in the strings of modalities.

FS4 is constructed by adding an S4 axiom and an FS4 axiom:

A 12 Lp $\rightarrow$ LLp

A 13 FMp $\rightarrow$ Mp - If *p* is found to be conceivable, then *p* is conceivable.[43]

to the axioms for FT. The formation and transformation rules remain the same and FS4 contains as theses all S4 or FT theses.

Among the theses of S4 that are useful in FS4 are those that follow.[44] Their English translations are not particularly interesting, but the theorems are needed to prove more interesting theorems regarding findings.

Theorem 18: Lp $\leftrightarrow$ LLp
Theorem 19: Mp $\leftrightarrow$ MMp
Theorem 20: LMp $\leftrightarrow$ LMLMp
Theorem 21: MLp $\leftrightarrow$ MLMLp
Theorem 22: MLMp $\rightarrow$ Mp.

Additional FS4 theorems include the following:

Theorem 23: FLp $\rightarrow$ Fp: If *p* is found to be conceptually necessary, *p* may be taken as found.
Theorem 24: MFp $\rightarrow$ Mp: If it is conceivable that *p* be found, *p* is conceivable.
Theorem 25: LFp $\rightarrow$ Fp: If it is conceptually necessary that *p* be found, *p* may be taken as found.
Theorem 26: FFp $\rightarrow$ Mp: If it is found that *p* is found, then *p* is conceivable.

Still other FS4 theorems follow. Their English translations are cumber-

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and FMLM—will be instances of the iterations considered and will be governed by the rules for them.

41. *See supra* notes 24-27 and accompanying text.

42. The transitive property may be expressed as: if H_1RH_2 and H_2RH_3, then H_1RH_3. Alternatively, if H_1RH_2, then $J(H_2)$ is a subset of $J(H_1)$.

43. To show that *FMp→Mp* holds in the model let f(FMp,H) = 1. Then:

$$\frac{\Sigma f(Mp,H')}{n} > \frac{1}{2.}$$

Since J(H) is not empty, there is an H' in J(H) such that f(Mp,H') = 1. Therefore, there is an H'' in J(H') such that f(p,H'') = 1. However, J(H') is a subset of J(H), so H'' is also in J(H) and f(p,H'') = 1. Hence f(Mp,H) = 1 and *FMp→Mp* holds in the model.

44. For proofs, see G. HUGHES & M. CRESSWELL, *supra* note 35, at 46-47.

some and the theorems are not particularly interesting in themselves, but they do provide for some simplification of more cumbersome expressions.

Theorem 27: LMFp → LMp
Theorem 28: MLFp → MFp
Theorem 29: LMLFp → MFp
Theorem 30: MLMFp → MFp
Theorem 31: FLMLp → MLp
Theorem 32: FMLMp → Mp
Theorem 33: FMLp → MLp
Theorem 34: FLMp → FMp

Unfortunately, only conditionals, rather than equivalences, result in FS4. Hence, only implications of, rather than reductions to, shorter iterations are accomplished for formulae containing *F*. However, it was not a result that could be avoided by a better choice of an FS4 axiom.[45]

IV. APPLICATIONS

Before turning to the analysis of collateral estoppel, it is interesting to note that the logic developed may also be used to derive additional findings from facts specially found, instructions and presumptions.

Example A: Suppose in a case there have been findings of (1) *p or q*, and (2) *v*. Suppose further that (3) *s* is conceptually necessary, (4) *not-p* is conceptually necessary, that (5) there has not been a finding of *not-r*, that (6) there is a presumption that *r*, and that (7) it is conceptually necessary that if *v*, then *t*. Suppose further that there is a jury instruction that (8) if the jury finds each of *q*, *r*, *s*, and *t*, then the jury must find *u*. Must the jury, based on (1)-(8), find *u*?

Analysis:[46] Since we have findings of (*p or q*) and that *not-p* is necessary, then by Theorem 12 we may infer that *q* is found.

A presumption of *r* is interpreted as the inference of *Fr* from

45. The model provides counterexamples to show that the conditionals proven in theorems 23-33 cannot be replaced by biconditionals, if the possible worlds semantics is to be a model for the system. *See infra* Appendix 2. The counterexamples in Appendix 2 are constructed to show also that the conditionals can not be "strengthened" to imply a "stronger" consequent.

It is possible to develop an extension of the logic S5. *See generally* C. LEWIS & C. LANGFORD, *supra* note 22, incorporating the operator F. The addition of *Mp→L Mp* and *M Fp→Fp* as axioms would cause any string of modalities to be equivalent to the last modality in the string. *See* G. HUGHES & M. CRESSWELL, *supra* note 35, at 46-49; Burgess, *Probability Logic*, 34 J. OF SYMBOLIC LOGIC 264 (1969). The resulting system would be equivalent to S5U developed by Burgess. The system would, however, contain unacceptable theses such as: "If *p* is conceivable, *p* may be taken as found." and "If it is possible that *p* is found, it is necessary that *p* is found." FS4 does not contain these objectionable theses and seems adequate for our purposes.

46. The analysis presented in the text is informal. Formal proofs of the conclusions reached in this example and in Example B may be found in Appendix 3.

$\sim F \sim r$ (the inference that *r* is found from the fact that *not-r* has not been found), and since there has not been a finding of *not-r*, *r* may be taken as found.

Axiom 7 states that if *s* is necessary, *s* may be taken as found.

Since we have a finding of *v* and it is conceptually necessary that *v* implies *t*, Axiom A 11 allows the inference that *t* is found.

Since *q*, *r*, *s* and *t* are all found, the jury instruction requires that *u* be found.

The proposed conclusion may be validly drawn in the logic.

We now turn to an example involving collateral estoppel.

Example B: Suppose prior litigation has (1) failed to find *p*. In that litigation there was an (2) instruction that if the jury found *q or r*, then it must find *p*, and there was a (3) presumption of *q*. May *not-q* be taken as established for collateral estoppel purposes?

Analysis: Since the instructions allow the inference of a finding of *p* from a finding of (*q or r*) and there was no finding of *p*, there was no finding of (*q or r*).

If (*q or r*) cannot be found, then by Theorem 13 neither *q* nor *r* may be found. In particular *q* is not found.

Under the presumption, if *not-q* were not found, *q* would be found. This would contradict the conclusion that *q* is not found, so *not-q* must have been found.

Admittedly, most collateral estoppel analysis will not be so simple. Seldom will the record, instructions and presumptions be so complete as to make the answer to the collateral estoppel question follow logically. The situation more often will be like the following.

Example C: Suppose a jury has failed to find *t* in a case in which there was a conclusive presumption of *not-r* and an instruction that if the jury found (*p and q*) and did not find *r*, they must find *t*. May a later court conclude that *p* was not found?

Any attempt to prove that *p* is not found ($\sim$Fp) within the logic will fail.[47] Yet, the logic may still be of value. An analysis may help provide insight into the case.

Analysis: Findings of (*p and q*) and a failure to find *r* will require that *t* be found. *T* was not found, so either *r* was found or (*p and q*) was not. There was a conclusive presumption of *not-r*, i.e., L$\sim$r. By Theorem 8 that implies a finding of *not-r* and then, by Theorem 9, *r* was not found. Hence, the failure to find *t* must be based on the failure to find (*p and q*). But even if both *p* and *q* were found that would not imply that (*p and q*) was found. Theorem 15 requires

47. The model may be used to construct a counterexample in which there is a world in which *t* is not found, *not-r* is necessary, and "if *p and q* is found and *r* is not found, then *t* is found" is necessary, yet in which *p* is found. Such a counter-example shows that $F \sim p$ should not be provable in the logic. One such counter-example is presented in Appendix 3.

> that one conjunct be found and the other be necessary, before the conjunction must be taken as found.

The analysis was valuable. Intuitively, it would appear that the court would simply have to decide that *q* was found before deciding that the failure to find *t* depended on a failure to find *p*. Rather, the analysis reveals that the court must conclude that *q* was conceptually necessary, before it may conclude that *p* was not found.[48]

Even when the logic presented here does not lead to a proof that something was or was not found, an analysis guided by the logic may clarify the problem. It may serve to identify the cause of the failure of logical proof. It makes clear the claims that the court must argue from a non-formal point of view, through an examination of the evidence and a determination of what the jury must or could have found, in order to come to a conclusion on a collateral estoppel question.

It is hoped that the logic developed here will help in attaining the "rationality" that is expected of collateral estoppel analysis. The rigor of a formal system has had such an effect wherever employed. The development of such a system for the analysis of fact finding should have a similar effect, and even if the systems are not employed formally, their study should help to clarify for the reader what form an informal analysis must approximate to be rational.

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48. If the instruction had instead been that if *p* was found and *q* was found and *r* was not found then *t* must be found, the first-blush analysis would have been correct. The court would only have to conclude that *q* was found before concluding that *p* was not found. On this issue see L. COHEN, *supra* note 16, at 58-67.

APPENDIX I: PROOFS OF THEOREMS

Proofs of theorems will consist of a series of well formed formulae the last of which is the thesis to be proved. Each formula in the series will either be an axiom, a previously proved theorem, formula derived from earlier formulae in the series, or a dischargeable hypothesis under the deduction theorem or indirect inference. The dischargeable hypothesis must actually be discharged before the proof is complete and any steps dependent on the dischargeable hypothesis may not be used after the hypothesis is discharged. Assumptions may also be included, but the resulting conclusions will be assumption-dependent and will not be theses. Each step in a proof will be justified by indicating its status as an axiom, assumption or theorem, or by indicating any transformation rules and previous steps involved. Lastly, since any thesis of the propositional calculus is also a thesis of FMn, rather than reprove each such thesis, they may be listed as steps and justified as "PC Thesis."

Theorem 8: Fp→F(pvq).

(1) Fp	Dis Hyp, DT
(2) p→(pvq)	PC Thesis
(3) L(p→(pvq))	2, Necessitation
(4) Fp&L(p→(pvq))	1, 3, CI
(5) (Fp&L(p→(pvq)))→F(pvq)	All (Uniform Substitution)
(6) F(pvq)	4, 5, Modus Ponens
(7) Fp→F(pvq)	DT, Dis 1

The justification for step (1) indicates a dischargeable hypothesis for use with the deduction theorem. The justification for step (7) indicates that the deduction theorem has been employed and the hypothesis in step (1) discharged.

Theorem 9: F~p→~Fp

(1) ~(F~p→~Fp)	Dis Hyp, II
(2) ~(F~p→~Fp)→(F~p&~~Fp)	PC Thesis (Uniform Substitution)
(3) F~p&~~Fp	1, 2, Modus Ponens
(4) ~~Fp↔Fp	PC Thesis (Uniform Substitution)
(5) F~p&Fp	3, 4 Substitution of Equivalents
(6) (F~p&Fp)→(Fp&F~p)	PC Thesis (Uniform Substitution)
(7) Fp&F~p	5, 6, Modus Ponens
(8) ~(Fp&F~p)	A8
(9) (Fp&F~p)&~(Fp&F~p)	7, 8, CI
(10) F~p→~Fp	II, Dis 1

The justification for steps (1) and (10) indicate indirect inference.

Theorem 10: Fq→F(p→q)

(1) Fq	Dis Hyp, DT
(2) Fq→F(qv~p)	Theorem 8 (Uniform Substitution)
(3) F(qv~p)	1, 2, Modus Ponens

(4) $(q \vee \sim p) \leftrightarrow (p \rightarrow q)$	PC Thesis
(5) $L((q \vee \sim p) \leftrightarrow (p \rightarrow q))$	4, Necessitation
(6) $F(p \rightarrow q)$	3, 5, Substitution of Strict Equivalents
(7) $Fq \rightarrow F(p \rightarrow q)$	DT, Dis 1

Theorem 11: $F(p\&q) \rightarrow F(p \vee q)$

(1) $F(p\&q)$	Dis Hyp, DT
(2) $F(p\&q) \rightarrow (Fp\&Fq)$	A9
(3) $Fp\&Fq$	1, 2, Modus Ponens
(4) Fp	3, CS
(5) $Fp \rightarrow F(p \vee q)$	Theorem 8
(6) $F(p \vee q)$	4, 5, Modus Ponens
(7) $F(p\&q) \rightarrow F(p \vee q)$	DT, Dis 1

Theorem 12: $(F(p \vee q)\&L \sim p) \rightarrow Fq$

(1) $F(p \vee q)\&L \sim p$	Dis Hyp, DT
(2) $F(p \vee q)$	1, CS
(3) $(p \vee q) \leftrightarrow (\sim p \rightarrow q)$	PC Thesis
(4) $L((p \vee q) \leftrightarrow (\sim p \rightarrow q))$	3, Necessitation
(5) $F(\sim p \rightarrow q)$	2, 4, Substitution of Strict Equivalents
(6) $L \sim p$	1, CS
(7) $L \sim p\&F(\sim p \rightarrow q)$	5, 6, CI
(8) $(L \sim p\&F(\sim p \rightarrow q)) \rightarrow Fq$	A10 (Uniform Substitution)
(9) Fq	7, 8, Modus Ponens
(10) $(F(p \vee q)\&L \sim p) \rightarrow Fq$	DT, Dis 1

Theorem 13: $\sim F(p \vee q) \rightarrow (\sim Fp\& \sim Fq)$

(1) $\sim(\sim F(p \vee q) \rightarrow (\sim Fp\& \sim Fq))$	DisHyp, II
(2) $\sim(\sim F(p \vee q) \rightarrow (\sim Fp\& \sim Fq)) \rightarrow (\sim F(p \vee q)\& \sim(\sim Fp\& \sim Fq))$	PC Thesis (Uniform Substitution)
(3) $\sim F(p \vee q)\& \sim(\sim Fp\& \sim Fq)$	1, 2, Modus Ponens
(4) $\sim F(p \vee q)$	3, CS
(5) $\sim(\sim Fp\& \sim Fq)$	3, CS
(6) $\sim(\sim Fp\& \sim Fq) \rightarrow (Fp \vee Fq)$	PC Thesis (Uniform Substitution)
(7) $Fp \vee Fq$	5, 6, Modus Ponens
(8) $Fp \rightarrow F(p \vee q)$	Theorem 8
(9) $Fq \rightarrow F(q \vee p)$	Theorem 8 (Uniform Substitution)
(10) $(q \vee p) \leftrightarrow (p \vee q)$	PC Thesis
(11) $L((q \vee p) \leftrightarrow (p \vee q))$	10, Necessitation
(12) $Fq \rightarrow F(p \vee q)$	9, 11, Substitution of Equivalents
(13) $(Fp \vee Fq) \rightarrow F(p \vee q)$	8, 12, IC
(14) $F(p \vee q)$	7, 13, Modus Ponens
(15) $F(p \vee q)\& \sim F(p \vee q)$	4, 14, CI
(16) $\sim F(p \vee q) \rightarrow (\sim Fp\& \sim Fq)$	II, Dis 1

Theorem 14: $L(p\to q)\to(Fp\to Fq)$

(1) $L(p\to q)$	Dis Hyp, DT
(2) Fp	Dis Hyp, DT
(3) $Fp\&L(p\to q)$	1, 2, CI
(4) $(Fp\&L(p\to q))\to Fq$	A 11
(5) Fq	3, 4, Modus Ponens
(6) $Fp\to Fq$	DT, Dis 2
(7) $L(p\to q)\to(Fp\to Fq)$	DT, Dis 1

Theorem 15: $(Lp\&Fq)\to F(p\&q)$

(1) $p\to(q\to(p\&q))$	PC Thesis
(2) $L(p\to(q\to(p\&q)))$	1, Necessitation
(3) $L(p\to(q\to(p\&q)))\to(Lp\to L(q\to(p\&q)))$	A6 (Uniform Substitution)
(4) $Lp\to L(q\to(p\&q))$	2, 3, Modus Ponens
(5) $L(q\to(p\&q))\to(Fq\to F(p\&q))$	Theorem 14 (Uniform Substitution)
(6) Lp	Dis Hyp, DT
(7) $L(q\to(p\&q))$	4, 6, Modus Ponens
(8) $Fq\to F(p\&q)$	5, 7, Modus Ponens
(9) $Lp\to(Fq\to F(p\&q))$	DT, Dis 6
(10) $(Lp\to(Fq\to F(p\&q)))\to((Lp\&Fq)\to F(p\&q))$	PC Thesis (Uniform Substitution)
(11) $(Lp\&Fq)\to F(p\&q)$	9, 10, Modus Ponens

Theorem 16: $Fp\to Mp$

(1) Fp	Dis Hyp, DT
(2) $\sim Mp$	Dis Hyp, II
(3) $L\sim p\leftrightarrow\sim M\sim\sim p$	Theorem 2 (Uniform Substitution)
(4) $\sim\sim p\leftrightarrow p$	PC Thesis
(5) $L\sim p\leftrightarrow\sim Mp$	3, 4, Substitution of Equivalents
(6) $\sim Mp\to L\sim p$	5, CS
(7) $L\sim p$	2, 6, Modus Ponens
(8) $L\sim p\to F\sim p$	A7 (Uniform Substitution)
(9) $F\sim p$	7, 8, Modus Ponens
(10) $Fp\&F\sim p$	1, 9, CI
(11) $\sim(Fp\&F\sim p)$	A8
(12) $(Fp\&F\sim p)\&\sim(Fp\&F\sim p)$	10, 11, CI
(13) Mp	II, Dis 2
(14) $Fp\to Mp$	DT, Dis 1

Theorem 17: $p\to Mp$

(1) $L\sim p\to\sim p$	A5 (Uniform Substitution)
(2) $(L\sim p\to\sim p)\to(p\to\sim L\sim p)$	PC Thesis (Uniform Substitution)
(3) $p\to\sim L\sim p$	1, 2, Modus Ponens
(4) $p\to Mp$	3, Substitution of Equivalents (Definition of "M")

Theorem 23: FLp→Fp

(1) Lp→p	A5
(2) L(Lp→p)	1, Necessitation
(3) L(Lp→p)→(FLp→Fp)	Theorem 14 (Uniform Substitution)
(4) FLp→Fp	2, 3, Modus Ponens

Theorem 24: MFp→Mp

(1) Fp→Mp	Theorem 16
(2) L(Fp→Mp)	1, Necessitation
(3) L(Fp→Mp)→(MFp→MMp)	Theorem 14 (Uniform Substitution)
(4) MFp→MMp	2, 3, Modus Ponens
(5) Mp↔MMp	Theorem 19
(6) MFp→Mp	4, 5, Substitution of Equivalents

Theorem 25: LFp→Fp

(1) LFp→Fp	A5 (Uniform Substitution)

Theorem 26: FFp→Mp

(1) Fp→Mp	Theorem 16
(2) L(Fp→Mp)	1, Necessitation
(3) L(Fp→Mp)→(FFp→FMp)	Theorem 14 (Uniform Substitution)
(4) FFp→FMp	2, 3, Modus Ponens
(5) FMp→Mp	A13
(6) (FFp→FMp)&(FMp→Mp)	4, 5, CI
(7) ((FFp→Fmp)&(FMp→Mp))→(FFp→Mp)	PC Thesis (Uniform Substitution)
(8) FFp→Mp	6, 7, Modus Ponens

Theorem 27: LMFp→LMp

(1) MFp→Mp	Theorem 24
(2) L(MFp→Mp)	1, Necessitation
(3) L(MFp→Mp)→(LMFp→LMp)	A6 (Uniform Substitution)
(4) LMFp→LMp	2, 3, Modus Ponens

Theorem 28: MLFp→MFp

(1) LFp→Fp	A5 (Uniform Substitution)
(2) L(LFp→Fp)	1, Necessitation
(3) L(LFp→Fp)→(MLFp→MFp)	Theorem 7 (Uniform Substitution)
(4) MLFp→MFp	2, 3, Modus Ponens

Theorem 29: LMLFp→MFp

(1) LMLFp	Dis Hyp, DT
(2) LMLFp→MLFp	A5 (Uniform Substitution)
(3) MLFp	1, 2, Modus Ponens
(4) LFp→Fp	A5 (Uniform Substitution)
(5) L(LFp→Fp)	4, Necessitation
(6) L(LFp→Fp)→(MLFp→MFp)	Theorem 7 (Uniform Substitution)

(7) MLFp→MFp	5, 6, Modus Ponens
(8) MFp	3, 7, Modus Ponens
(9) LMLFp→MFp	DT, Dis 1

Theorem 30: MLMFp→MFp

(1) MLMFp	Dis Hyp, DT
(2) LMFp→MFp	A5 (Uniform Substitution)
(3) L(LMFp→MFp)	2, Necessitation
(4) L(LMFp→MFp)→(MLMFp→MMFp)	Theorem 7 (Uniform Substitution)
(5) MLMFp→MMFp	3, 4, Modus Ponens
(6) MMFp	1, 5, Modus Ponens
(7) MFp↔MMFp	Theorem 19 (Uniform Substitution)
(8) MMFp→MFp	7, CS (Definition of "↔")
(9) MFp	6, 8, Modus Ponens
(10) MLMFp→MFp	DT, Dis 1

Theorem 31: FLMLp→MLp

(1) FLMLp	Dis Hyp, DT
(2) FLMLp→FMLp	Theorem 23 (Uniform Substitution)
(3) FMLp	1, 2, Modus Ponens
(4) FMLp→MLp	A13 (Uniform Substitution)
(5) MLp	3, 4, Modus Ponens
(6) FLMLp→MLp	DT, Dis 1

Theorem 32: FMLMp→Mp

(1) FMLMp→MLMp	A13 (Uniform Substitution)
(2) MLMp→Mp	Theorem 22
(3) (FMLMp→MLMp)&(MLMp→Mp)	1, 2, CI
(4) ((FMLMp→MLMp)&(MLMp→Mp))→(FMLMp→Mp)	PC Thesis (Uniform Substitution)
(5) FMLMp→Mp	3, 4, Modus Ponens

Theorem 33: FMLp→MLp

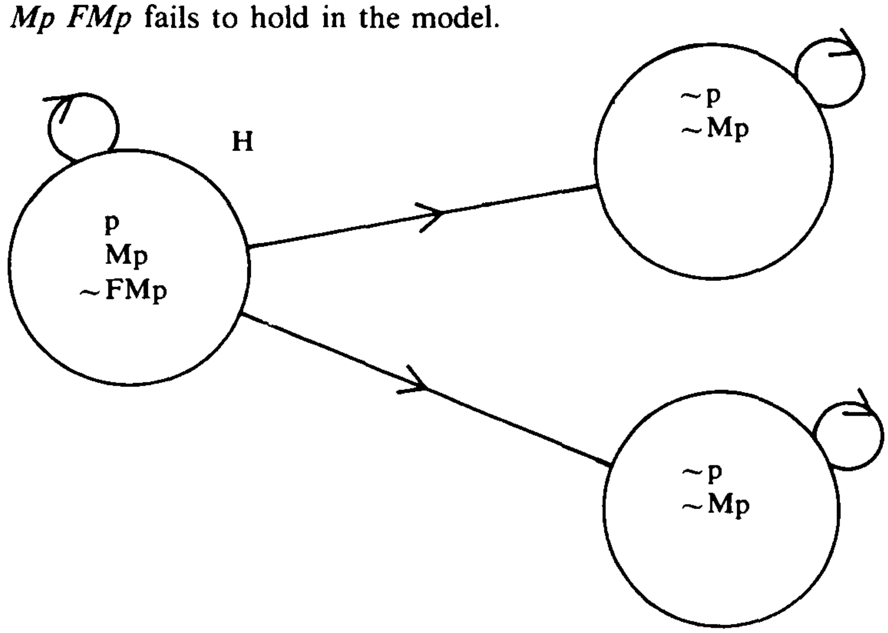
(1) FMLp→MLp	A13 (Uniform Substitution)

Theorem 34: FLMp→FMp

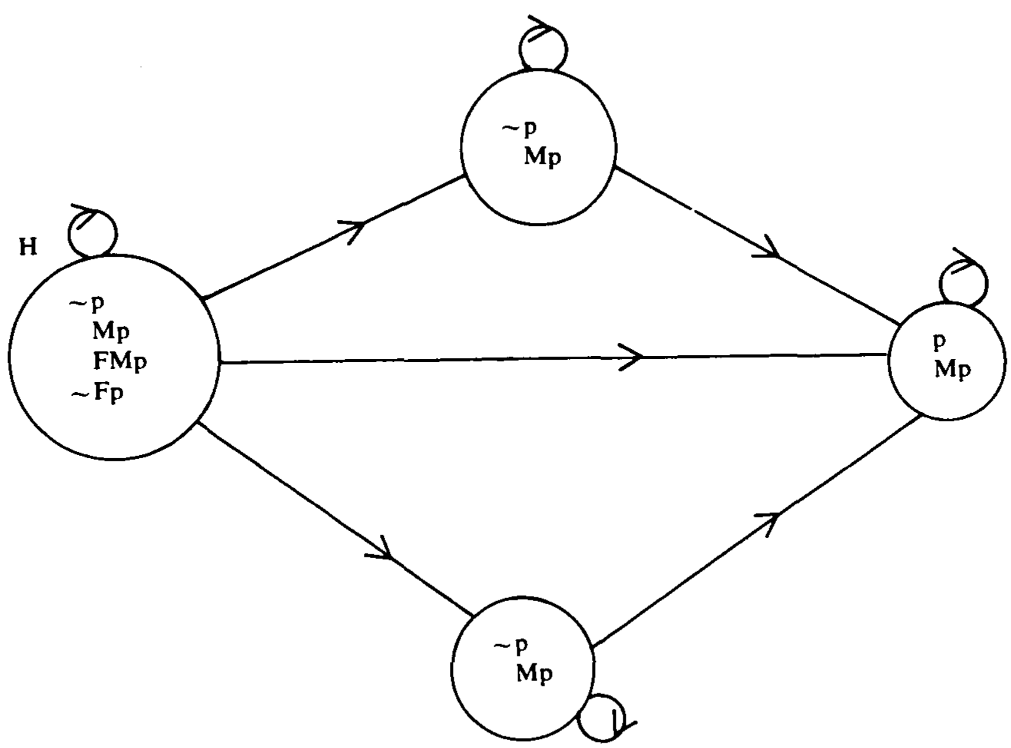
(1) LMp→Mp	A5 (Uniform Substitution)
(2) L(LMp→Mp)	1, Necessitation
(3) L(LMp→Mp)→(FLMp→FMp)	Theorem 14 (Uniform Substitution)
(4) FLMp→FMp	2, 3, Modus Ponens

APPENDIX 2: FS4 COUNTEREXAMPLES

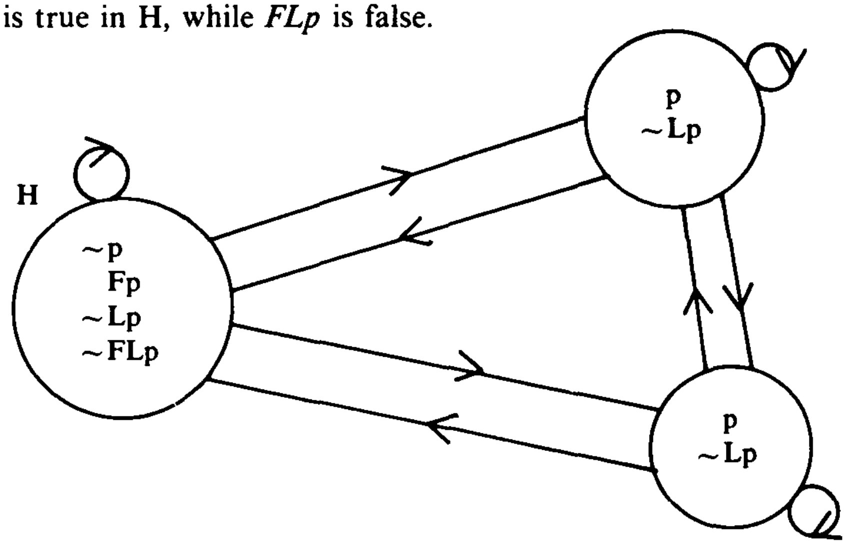
The figure below shows that Axiom A13 should not have been strengthened to a biconditional. *Mp* is true in H but *FMp* is false in H, so *Mp FMp* fails to hold in the model.



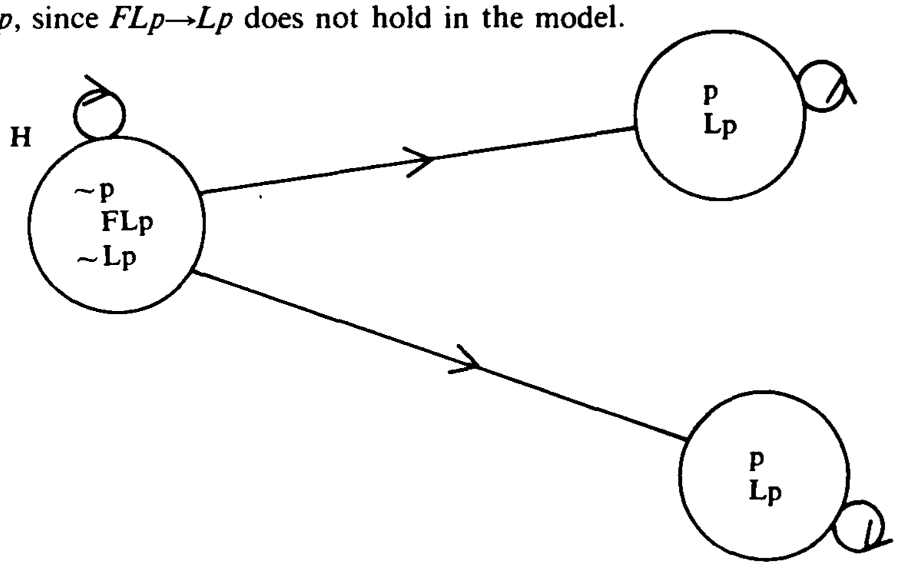
The following figure shows that Axiom A13 cannot be strengthened to *FMp→Fp*. *FMp* is true in H but *Fp* is false.



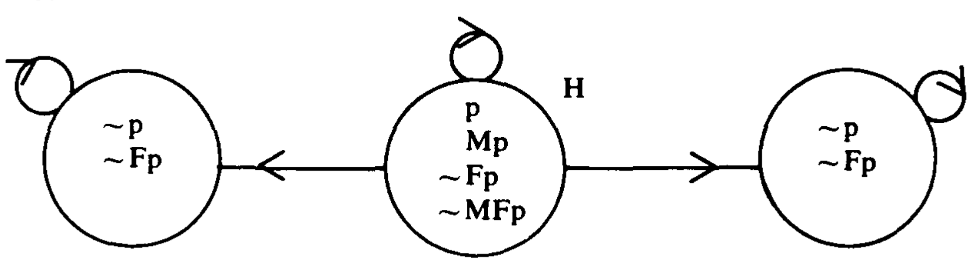
The converse of Theorem 23, *Fp→FLp*, does not hold in the model. *Fp* is true in H, while *FLp* is false.



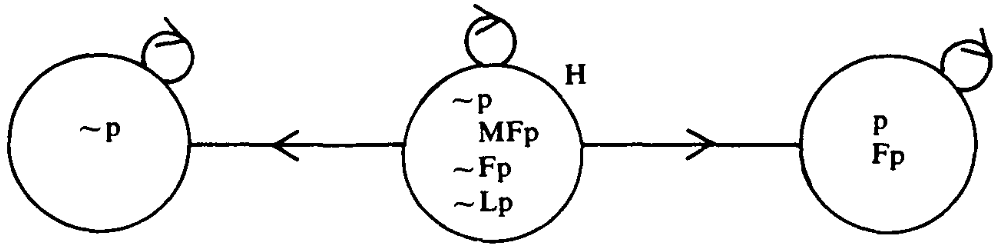
Nor may Theorem 23 be strengthened by changing the consequent to *Lp*, since *FLp→Lp* does not hold in the model.



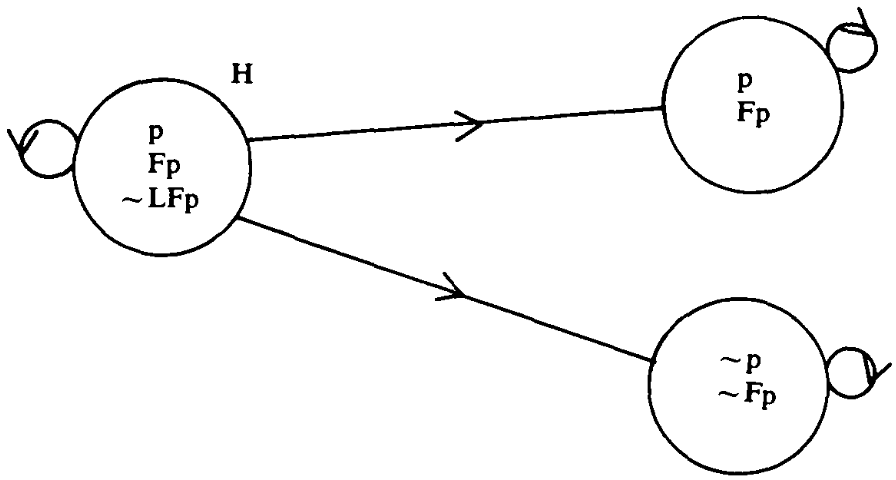
The converse of Theorem 24, *Mp→MFp*, does not hold in the model, as the following counterexample shows. *Mp* is true in H, but *MFp* is not.



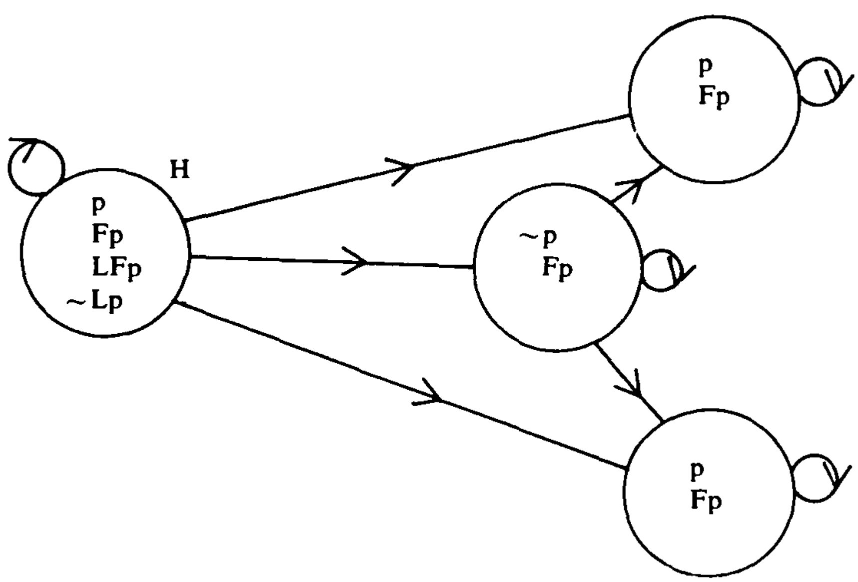
The theorem may also not be strengthened to *MFp→Fp*, and hence not to *MFp→Lp*, as is shown by the figure below.



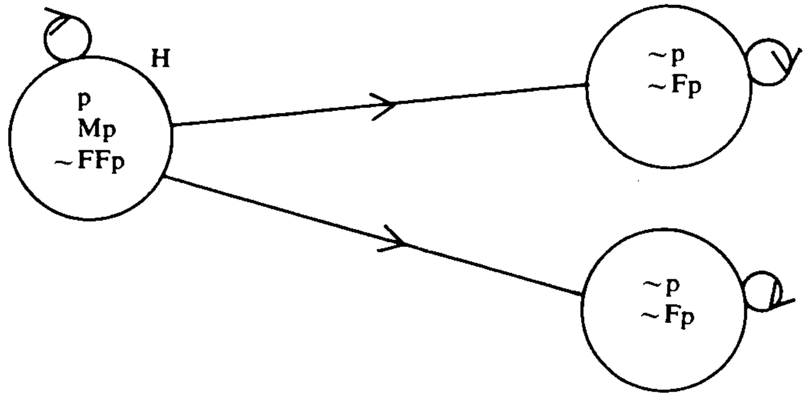
The following diagram shows that the converse of Theorem 25, *Fp→LFp*, does not hold in the model.



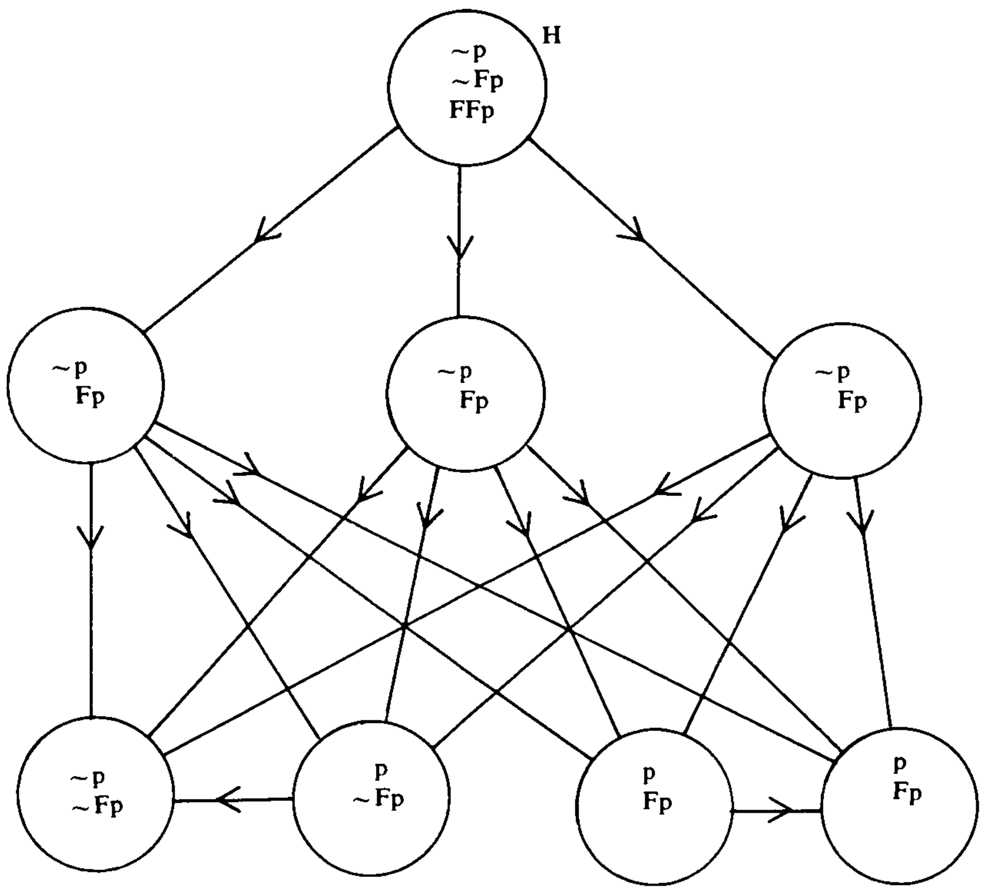
The theorem may also not be strengthened to LFP→LP, as is shown below.



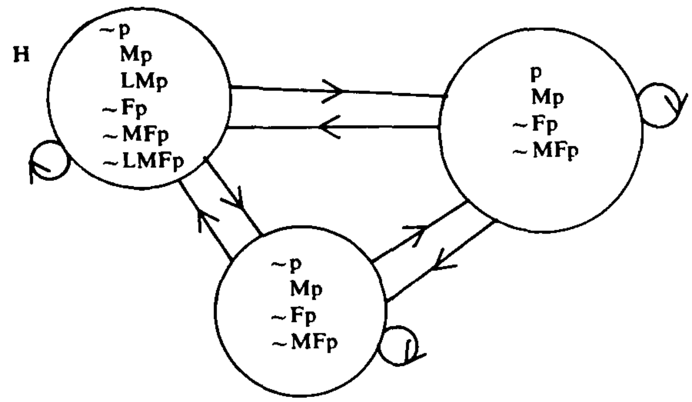
The converse of Theorem 26, *Mp→FFp*, also fails in the model.



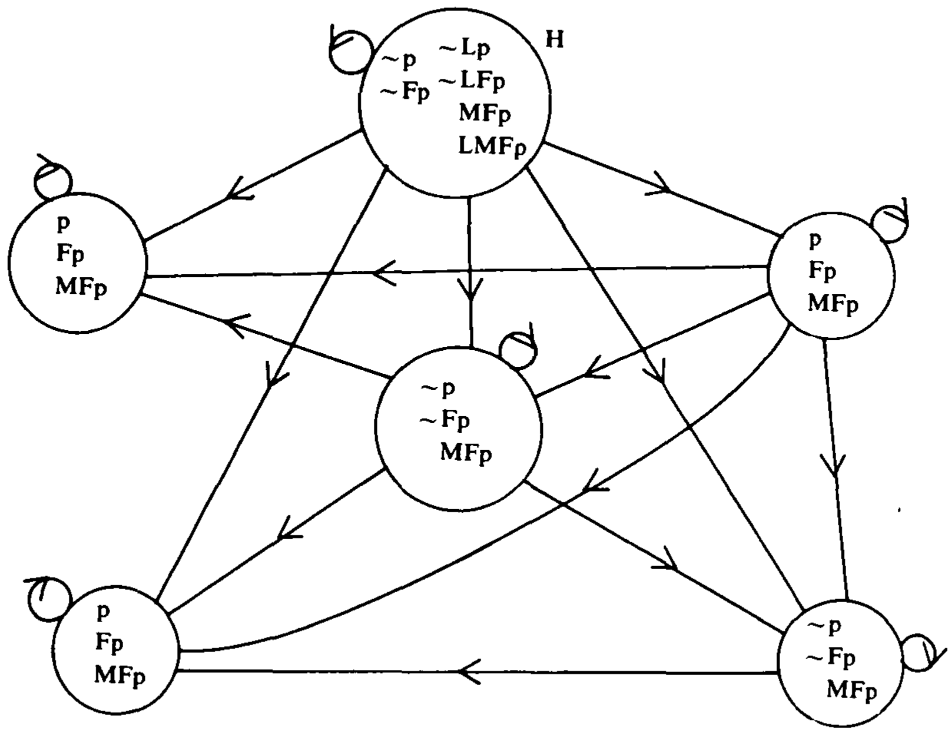
The theorem may also not be strengthened to *FFp→Fp*, as shown by the figure below. The reflexive loops have been omitted but each world does have access to itself. There is also an omitted arrow from H to each of the bottom row of worlds, as required by transitivity.



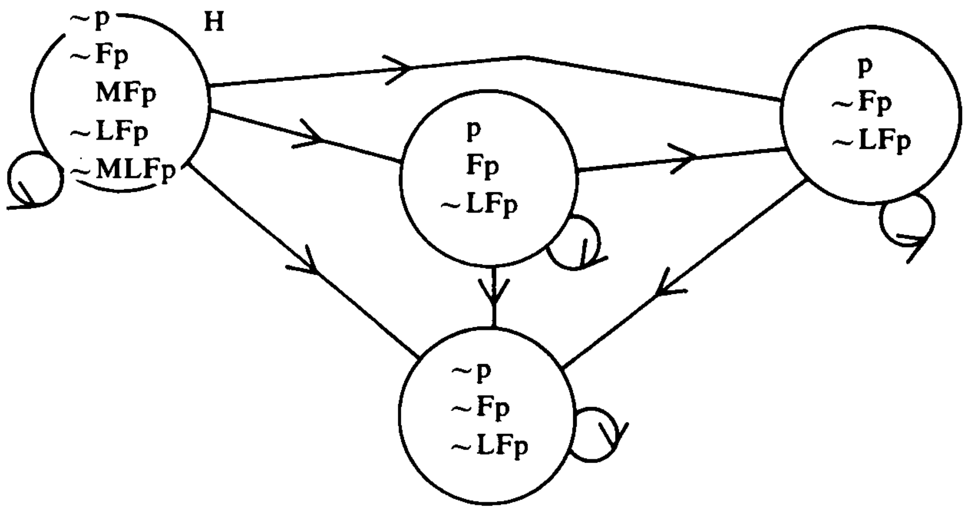
The converse of Theorem 27, *LMp→LMFp*, should not be expected to hold.



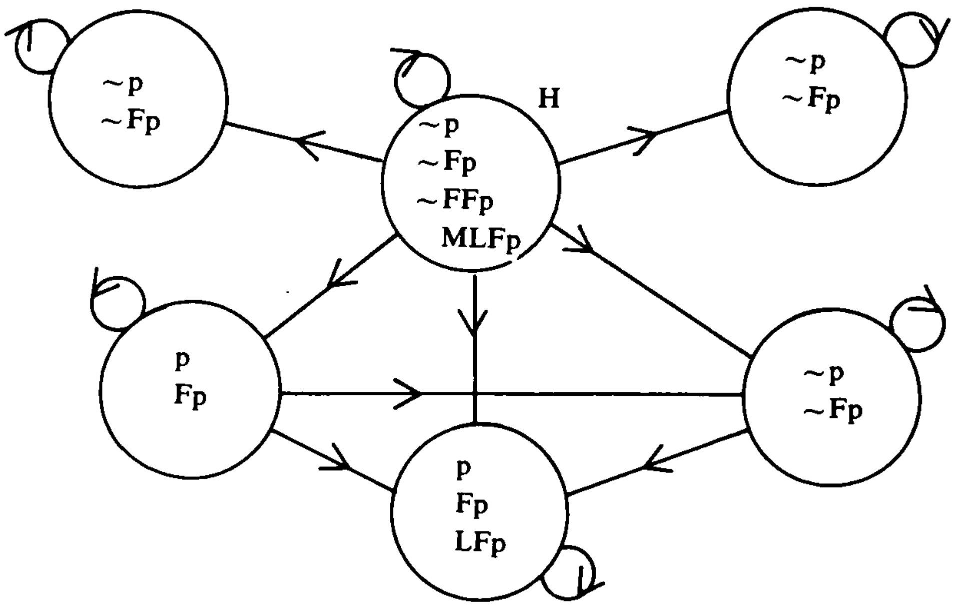
Another counterexample shows that *LMFp→LMp* cannot be strengthened by changing the consequent of the conditional to *Lp* or *LFp* or even *Fp*, if we are to remain true to the model.



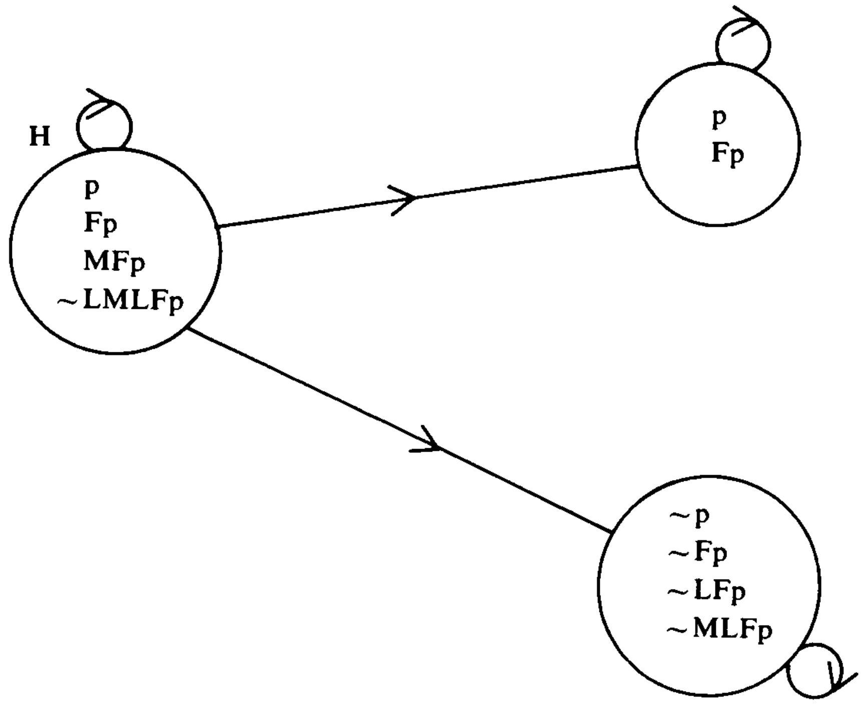
As the following figure shows, the converse of Theorem 28, *MFp→MLFp*, fails to hold in the model.



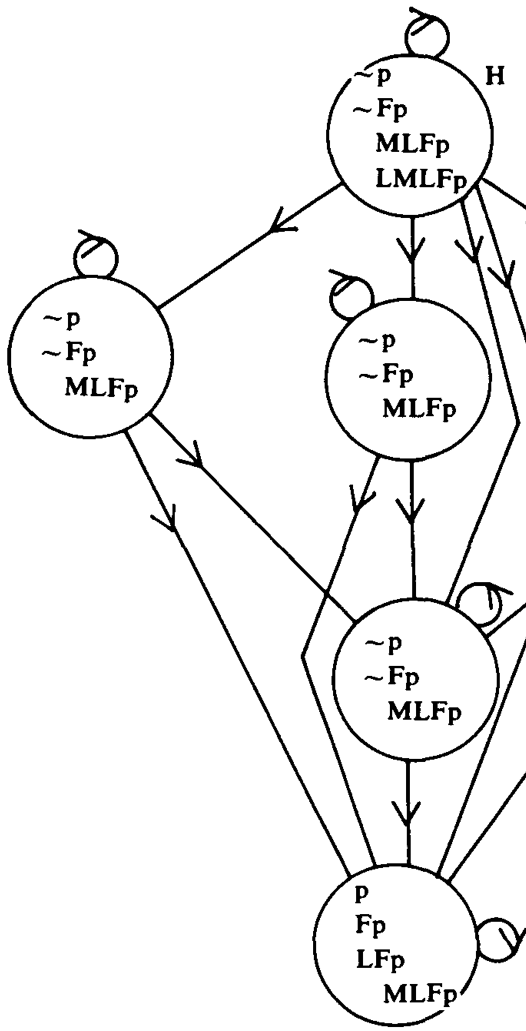
The theorem may also not be strengthened to *MLFp→Fp* or *MLFp→FFp*.



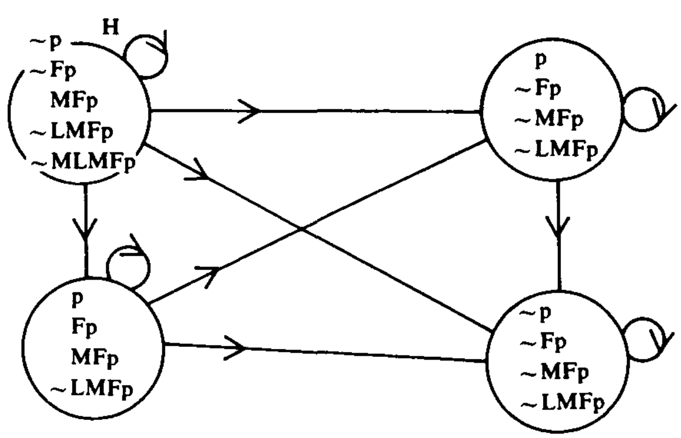
The converse of Theorem 29, *MFp→LMLFp*, also fails in the model, as is shown below.



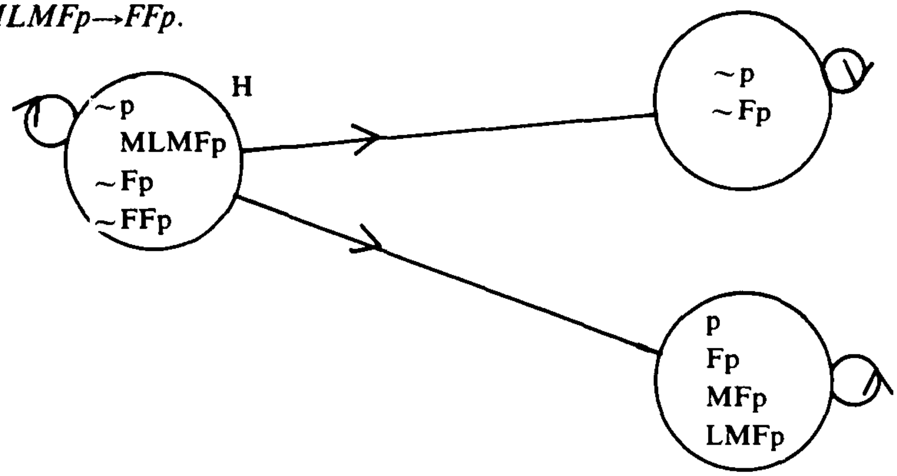
Nor may Theorem 29 be strengthened to *LMLFp→Fp*.



The following figure shows that the converse of Theorem 30, *MFp→MLMFp*, fails to hold in the model.



Nor may the theorem be strengthened to either *MLMFp→Fp* or *MLMFp→FFp*.



Theorems 31-34 involve instances of *FLp* or *FMp* and a strengthening of those theorems would depend on a strengthening of the FS4 axiom or of earlier theorems — strengthening that would not hold in the model.

APPENDIX 3: PROOFS AND COUNTEREXAMPLE FOR SECTION IV: APPLICATIONS

Example A:

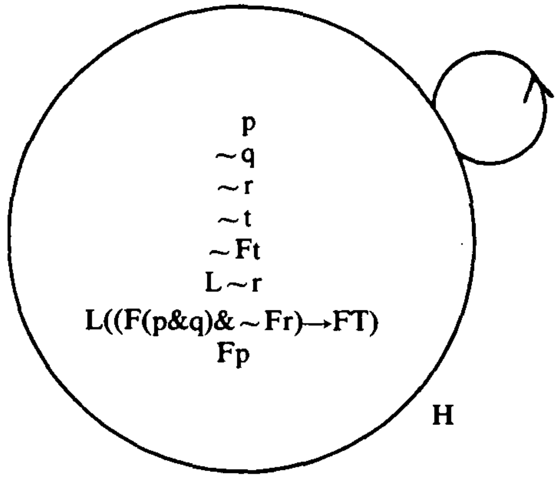
(1) F(pvq)	Assumption
(2) Fv	Assumption
(3) Ls	Assumption
(4) L∼p	Assumption
(5) ∼F∼r	Assumption
(6) ∼F∼r→Fr	Assumption
(7) L(v→t)	Assumption
(8) L((((Fq&Fr)&Fs)&Ft)→Fu)	Assumption
(9) F(pvq)&L∼p	1, 4, CI
(10) (F(pvq)&L∼p)→Fq	Theorem 12
(11) Fq	9, 10, Modus Ponens
(12) Fr	5, 6, Modus Ponens
(13) Ls→Fs	A7 (Uniform Substitution)
(14) Fs	3, 13, Modus Ponens
(15) Fv&L(v→t)	2, 7, CI
(16) (Fv&L(v→t))→Ft	A11 (Uniform Substitution)
(17) Ft	15, 16, Modus Ponens
(18) L((((Fq&Fr)&Fs)&Ft)→Fu)→ ((((Fq&Fr)&Fs)&Ft)→Fu)	A5 (Uniform Substitution)
(19) (((Fq&Fr)&Fs)&Ft)→Fu	8, 18, Modus Ponens
(20) (((Fq&Fr)&Fs)&Ft	11, 12, 14, 17, CI, CI, CI
(21) Fu	19, 20, Modus Ponens

Example B:

(1) ∼Fp	Assumption
(2) L(F(qvr)→Fp)	Assumption
(3) ∼F∼q→Fq	Assumption
(4) L(F(qvr)→Fp)→(F(qvr)→Fp)	A5 (Uniform Substitution
(5) F(qvr)→Fp	2, 4, Modus Ponens
(6) (F(qvr)→Fp)&∼Fp	1, 5, CI
(7) ((F(qvr)→Fp)&∼Fp)→∼F(qvr)	PC Thesis (Uniform Substitution)
(8) ∼F(qvr)	6, 7, Modus Ponens
(9) ∼F(qvr)→(∼Fq&∼Fr)	Theorem 13 (Uniform Substitution)
(10) ∼Fq&∼Fr	8, 9, Modus Ponens
(11) ∼Fq	10, CS
(12) (∼F∼q→Fq)→(∼Fq→F∼q)	PC Thesis (Uniform Substitution)
(13) ∼Fq→F∼q	3, 12, Modus Ponens
(14) F∼q	11, 13, Modus Ponens

Example C:

A single world model in which *p* is true and *q*, *r* and *t* are false provides such a counterexample.



Since *t* is false in H and H has access only to itself, *t* is not found in H. Since *r* is false in H, and again H has access only to itself, *~r* is necessary in H. The third assumption, the jury instruction, is more difficult to derive but follows from the falsity of *q*. Since *q* is false, so is *p&q*. Since *p&q* is false in H and H has access only to itself, *p&q* is not found in H. Therefore, it cannot be that *p&q* is found and *r* is not found. Since a false antecedent makes a conditional true, *((F(p&q)&~Fr)→Ft* is true in H, and since H has access only to itself, the conditional is necessarily true in H. Hence, all the assumptions are true in H, yet since *p* is true in H and H has access only to itself, *p* is found in H.